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Spatial distribution of persistent sites

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Abstract. We study the distribution of persistent sites (sites unvisited by particles A) in the one-dimensional $A+A \to \emptyset$ reaction–diffusion model. We define the *empty intervals* as the separations between adjacent persistent sites, and study their size distribution $n(k,t)$ as a function of interval length k and time t. The decay of persistence is the process of irreversible coalescence of these empty intervals, which we study analytically under the independent interval approximation (IIA). Physical considerations suggest that the asymptotic solution is given by the dynamic scaling form $n(k,t) = s^{-2} f(k/s)$ with the average interval size $s \sim t^{1/2}$. We show under the IIA that the scaling function $f(x) \sim x^{-\tau}$ as $x \to 0$ and decays exponentially at large x. The exponent τ is related to the persistence exponent θ through the scaling relation $\tau = 2(1-\theta)$. We compare these predictions with the results of numerical simulations. We determine the two-point correlation function $C(r,t)$ under the IIA. We find that for $r \ll s$, $C(r,t) \sim r^{-\alpha}$ where $\alpha = 2 - \tau$, in agreement with our earlier numerical results.

1. Introduction

The persistence of fluctuations in stochastic processes has been an important topic of study in recent times [1]. Of primary interest in this context is the persistence probability $P(t)$, which is the probability that a given stochastic variable $\phi(t)$ with zero mean retains its sign during the time interval $[0,t]$. A power-law decay $P(t) \sim t^{-\theta}$ is found in many systems of physical interest. Consequently, much effort has gone into the calculation of the new exponent θ and studying its properties. There have also been some experimental studies of the persistence exponent θ. These include coarsening dynamics of breath figures [2], soap froth [3] and twisted nematic liquid crystals [4].

A particularly important class of systems whose persistence behaviour has been investigated are spatially extended systems with a stochastic field $\phi(\boldsymbol{x},t)$ at each lattice site $\boldsymbol{x}$. The time evolution of $\phi(\boldsymbol{x},t)$ is coupled to that of its neighbouring sites. $\phi(\boldsymbol{x},t)$ could be, for instance, an Ising spin [5,6], a phase ordering field [7], a diffusing field [8] or the height of a fluctuating interface [9]. The persistence probability $P(t)$ is then the fraction of sites $\boldsymbol{x}$ where $\phi(\boldsymbol{x},t)$ has not flipped sign by time t. Recently it was observed that the set of persistent sites forms a fractal and the time evolution of their spatial correlations obeys dynamic scaling [10]. The purpose of this paper is the investigation of these spatial correlations.

For concreteness, we study the one-dimensional $A + A \to \emptyset$ model. Our primary motivation for this choice is the simplicity of the dynamics of the model, which makes an analytic approach possible. In addition, this model is closely related to the $d = 1$ Glauber–Ising model, which is perhaps the only non-trivial model where θ is known exactly. We study the distribution of the separations between nearest-neighbour pairs of persistent sites. We

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call this the empty interval distribution $n(k, t)$, defined as the number of occurrences where consecutive persistent sites are separated by distance k at time t.

In this paper, we study the time evolution of the size distribution $n(k, t)$ of these empty intervals. Persistence decay is identified with the irreversible coalescence of these intervals. The paper is organized as follows. In the next section we write a rate equation for the coalescence of these intervals under the independent interval approximation (IIA) that the lengths of adjacent intervals are uncorrelated. We give phenomenological arguments about the asymptotically relevant dynamical length scale as well as the coalescence probability. These arguments, combined with the rate equation, give the dynamic scaling behaviour of $n(k, t)$ at late times t. We compare our predictions with numerical results. In section 3, we use the IIA to predict the two-point correlations in the distribution of persistent sites. The predictions are found to be in agreement with recent numerical results, showing that the IIA is valid.

2. The empty interval distribution

In the $A + A \rightarrow \emptyset$ model, a set of particles is distributed at random on the lattice with average density n_0. Over one time step, all the particles make an attempted jump to either of the neighbouring sites with some probability D. If two particles meet each other, both disappear from the lattice. In one dimension, the density of particles decays with time as $n(t) \sim (8\pi Dt)^{-\frac{1}{2}}$ as $t \rightarrow \infty$ [11]. Persistent sites in the $A + A \rightarrow \emptyset$ model at any time t are defined as the sites which remain unvisited by any diffusing particle throughout the time interval $[0 : t]$. Empty intervals (which we will call 'Interval' for simplicity henceforth) are defined as the separations between two consecutive persistent sites. By definition, an Interval cannot contain a persistent site, although it may contain one or more diffusing particles A. The total number (per site) of Intervals of length k at time t is denoted by $n(k, t)$ and is called the empty interval distribution.

To start with, the particles are put randomly on the lattice so that $n(k, t = 0) = n_0^2(1 - n_0)^k \sim \mathrm{e}^{-\lambda k}$ where $\lambda = -\log(1 - n_0)$. With time, the particles diffuse on the lattice, making the sites non-persistent. $n(k, t)$ evolves satisfying the following normalization conditions. If $I_m(t) = \sum_k k^m n(k, t) \approx \int_1^\infty n(s, t)s^m \, \mathrm{d}s$ is the mth moment of the distribution, then

$$\begin{aligned} I_0(t) &= P(t) \sim t^{-\theta} \\ I_1(t) &= 1 \\ I_2(t) &\equiv s(t). \end{aligned} \tag{1}$$

The first condition follows from the definition of $n(k, t)$, the second one implies length conservation and the third condition gives the mean interval size $s(t)$. The probability distribution of interval lengths is $p(k, t) = \frac{n(k,t)}{\sum_k n(k,t)} = P(t)^{-1} n(k, t)$ so that $\sum_k p(k, t) = 1$.

Two neighbouring Intervals can coalesce when the persistent site between them is destroyed by a diffusing particle at the boundary of either of the Intervals . Note that this coalescence process is irreversible. For simplicity, we consider only binary coalescence in a single time step where two adjacent Intervals of lengths k_1 and k_2, separated by a persistent site, coalesce and form a new Interval of length $k_1 + k_2$ when the persistent site is 'killed' by a particle (figure 1). To study this process analytically, we invoke a mean-field approximation—the lengths of adjacent Intervals are treated as uncorrelated random variables with probability distribution $p(k, t)$. This is the IIA, which has been used to study a variety of problems in one dimension [12, 13].

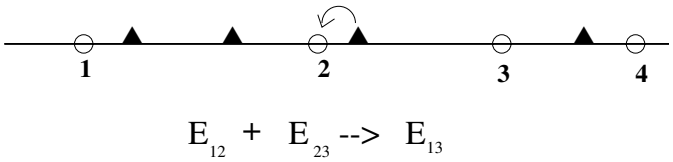

Figure 1. In the picture, white circles are persistent sites (numbered 1, 2, 3, ...) and dark triangles are diffusing particles. Two Intervals E_{12} and E_{23} are shown to merge together to give a new Interval E_{13} when the persistent site 2 at the boundary is killed by a diffusing particle.

2.1. Rate equation for interval coalescence

Assuming that IIA is valid, the time evolution of $n(k,t)$ is given by the rate equation

$$\frac{\partial n(k,t)}{\partial t} = \frac{1}{2}\sum_{m=1}^{k-1} n(m,t)\,p(k-m,t)\,K(m,k-m,t) - n(k,t)\sum_{m=1}^{\infty} p(m,t)\,K(m,k,t) \tag{2}$$

where $K(m_1, m_2, t)$ is the probability that two adjacent Intervals of lengths m_1 and m_2 coalesce at time t. The first term in equation (2) represents the increase in number of Intervals of size k through coalescence of smaller intervals, while the second term is the loss term representing the decrease in number when Intervals of size k merge with other Intervals.

To solve the above equation for $n(k,t)$, one needs to know the form of the reaction kernel $K(m_1, m_2, t)$. The process of coalescence of Intervals involves the destruction of the persistent site in between them by a particle, which can originate from either of the Intervals. So, quite generally,

$$K(m_1, m_2, t) = Q(m_1, t) + Q(m_2, t) \tag{3}$$

where $Q(m,t)$ is the fraction of intervals of size m which is destroyed at time t. $Q(m,t)$ satisfies the following condition by definition:

$$\sum_m n(m,t)\,Q(m,t) = -\frac{\partial P(t)}{\partial t} = \frac{\theta}{t}P(t) \tag{4}$$

where we have made use of the fact that $P(t) \sim t^{-\theta}$.

The form of $Q(m,t)$ can be argued for in the following way. An Interval of length m at time t can contain a particle anywhere inside it only if the interval length is at least of the order of the diffusive scale $\sqrt{Dt}$. That is, $Q(m,t) \simeq 0$ for $m \ll \sqrt{Dt}$. It is also known that the particle distribution is correlated over length scales $r \ll \sqrt{Dt}$, wheras it is completely random over $r \gg \sqrt{Dt}$ [13]. So we expect that for $m \gg \sqrt{Dt}$, $Q(m,t) \to \alpha(t)$, independent of m. These physical considerations lead us to suggest the following dynamic scaling form for $Q(m,t)$:

$$Q(m,t) = \alpha(t)\beta\left(\frac{m}{\sqrt{Dt}}\right) \tag{5}$$

where the function $\beta(x)$ is expected to have a sigmoidal form, i.e. $\beta(x) = 0$ for $x \ll 1$ and $\beta(x) \to 1$ for $x \gg 1$. The function $\alpha(t)$ will be determined later.

2.2. Dynamic scaling

We assume that, at asymptotic times, the distribution $n(k,t)$ is characterized by a single dynamic length scale $s(t)$. We note that there are two relevant length scales in the problem. The first is the diffusive scale $\mathcal{L}_D(t) \sim \sqrt{Dt}$ entering the scaling form equation (5) for the

coalescence probability. On the other hand, the inverse of the persistent fraction $P(t)$ is also a length scale, which we shall call the persistence scale, denoted by $\mathcal{L}_p(t) \sim t^{\theta}$. The asymptotic behaviour is expected to be dominated by the larger of the two, i.e. the diffusive scale $\mathcal{L}_D(t)$ in the present case (since $\theta < \frac{1}{2}$).

We now invoke the dynamic scaling ansatz, i.e. $n(k,t) \propto f(\frac{k}{s})$ with

$$s \sim t^{1/z} \qquad z = 2. \tag{6}$$

From the length conservation condition given by the second part of equation (1) it follows that the prefactor is $\sim s^{-2}$. Thus, the scaling solution for $n(k,t)$ is written in the form

$$n(k,t) = s(t)^{-2} f\left(\frac{k}{s(t)}\right). \tag{7}$$

Substituting equations (5) and (7) in (4), we find $\alpha(t)$.

$$\alpha(t) = \frac{\theta}{t}\frac{s(t)P(t)}{B} \tag{8}$$

where $B = \int_0^{\infty} \beta(x) f(x)\,\mathrm{d}x$. Substituting equation (7) in the normalization conditions equation (1), we find the following conditions on the scaling function:

$$\int_{s^{-1}}^{\infty} f(x)\,\mathrm{d}x = sP(t) \qquad \int_0^{\infty} f(x)x\,\mathrm{d}x = 1. \tag{9}$$

In the first integral, the lower limit is set as $s(t)^{-1}$ to take care of any possible small argument divergence.

Substituting equations (5)–(8) in (2), we find the following equation for the scaling function $f(x)$:

$$\frac{\eta}{z}\frac{\partial f}{\partial \eta} = -\frac{\theta}{B}\int_{s(t)^{-1}}^{\frac{\eta}{2}} f(x)f(\eta - x)[\beta(x) + \beta(\eta - x)]\,\mathrm{d}x - \left[\frac{2}{z} - \theta - \frac{\theta}{B}s(t)P(t)\beta(\eta)\right] f(\eta) \tag{10}$$

where the scaling variable $\eta = \frac{k}{s(t)}$.

Case 1. $\eta \ll 1$.

For $\eta \ll 1$, all $\beta(x) \simeq 0$ for $x \leqslant \eta$. This case corresponds to small Intervals, i.e. those which are not large enough to contain a diffusing particle until time t. In this case, the equation reduces to $\eta\frac{\partial f}{\partial \eta} = -(2 - z\theta) f(\eta)$, which has the solution $f(\eta) \sim \eta^{-\tau}$, where the new exponent τ is related to θ through the scaling relation:

$$\tau = 2 - z\theta. \tag{11}$$

From equation (7) this implies that for $k \ll s$, $n(k,t) \sim t^{-\theta}k^{-\tau}$. For the model under consideration here, θ is known exactly to be $\frac{3}{8}$ [6], which gives $\tau = \frac{5}{4}$.

Case 2. $\eta \gg 1$.

For general values of η, $\beta(\eta)$ is non-zero, and, because $\tau > 1$, the first integral diverges near $x = 0$ as $x^{-(\tau-1)}$. There is another divergence in the last term, of the form $t^{1/z-\theta}$. It can be shown that this term can be exactly cancelled by the divergent part of the first integral.

After carrying out this 'regularization' (details to be found in appendix A) and putting $z = 2$ in equation (10) the equation for the scaling function $f(\eta)$ stands as

$$\frac{\eta}{2}\frac{\partial f}{\partial \eta} = -\frac{\theta}{B}\int_0^{\frac{\eta}{2}} f(x)f(\eta - x)[\beta(x) + \beta(\eta - x) - \beta(\eta)]\,\mathrm{d}x - \frac{\theta}{B}\beta(\eta)\int_0^{\frac{\eta}{2}} f(x) \times [f(\eta - x) - f(\eta)]\,\mathrm{d}x - \left[1 - \theta - \frac{\theta}{B}\beta(\eta)\int_{\frac{\eta}{2}}^{\infty} f(x)\,\mathrm{d}x\right] f(\eta). \tag{12}$$

A general solution of this equation requires the knowledge of the detailed form of the scaling function $\beta(\eta)$. However, for large values of η where $\beta(\eta) \simeq 1$, one can simplify this equation. We define the point η^* as sufficiently large that, for $\eta \geqslant \eta^*$, $\beta(x) = 1$ within the limits of accuracy required. Without any loss of generality, one can put $\eta^* = 1$ by rescaling the length scale $s(t)$ accordingly. For $\eta \geqslant 1$, we define $f(\eta) \equiv h(\eta)$, whose equation is

$$\frac{\eta}{2}\frac{\partial h}{\partial \eta} = -\frac{\theta}{B}\left[2\int_1^{\frac{\eta}{2}} h(x)h(\eta - x)\,\mathrm{d}x + \int_0^1 f(x)[h(\eta - x) - h(\eta)]\,\mathrm{d}x\right] - (1 - 2\theta)h(\eta). \tag{13}$$

This equation has a solution of the form $h(\eta) = G\mathrm{e}^{-\lambda\eta}$ as can be shown by direct substitution. The constants G and λ are related through the relations

$$\lambda B = 2\theta G \tag{14}$$

and

$$\lambda + 2\theta = 1 + \frac{\theta}{B}F(\lambda) \tag{15}$$

where $F(\lambda) = \int_0^1 f(x)[\mathrm{e}^{\lambda x}(1 + \beta(x)) - 1]\,\mathrm{d}x$ and

$$B = \int_0^1 f(x)\beta(x)\,\mathrm{d}x + \frac{G}{\lambda}\mathrm{e}^{-\lambda} \tag{16}$$

by definition. Equations (14)–(16) formally give the constants λ and G. However, the actual evaluation of these constants requires the knowledge of the function $f(x)$ in the entire range $[0 : 1]$ (and not just near $x = 0$, where $f(x) \sim x^{-\tau}$), which, in turn, is possible only if the detailed form of $\beta(x)$ is known. Hence we will restrict ourselves to showing that the parameter $\lambda > 0$, which is required for the solution to be physically reasonable.

In equation (16), we note that $B \geqslant \frac{G}{\lambda}\mathrm{e}^{-\lambda}$, depending on how sharply $\beta(x)$ rises near $x = 1$. The equality holds for the step function $\beta(x) = \Theta(x - 1)$ where $\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x \geqslant 0$. After using this inequality in equation (14), we find that $\lambda \geqslant -\log(2\theta)$. Since $\theta < \frac{1}{2}$, it follows that $\lambda > 0$.

2.3. Numerical results

We determine the distribution $n(k, t)$ numerically by simulating $A + A \rightarrow \emptyset$ model on a one-dimensional lattice of size $N = 10^5$ with periodic boundary conditions. Particles are initially distributed at random on the lattice with some average density n_0, and their positions are sequentially updated—each particle is made to move one step in either direction with probability $D = \frac{1}{2}$. When two particles meet each other, both are removed from the lattice. The time evolution is observed up to 10^5 Monte Carlo steps (1 MC step is counted after all the particles in the lattice have been touched once). The simulation is repeated for several random starting configurations of the particles for any particular initial density and we repeat the entire simulation for four different initial densities n_0. For any n_0, we determine the number of intervals of length k (per site) at time t.

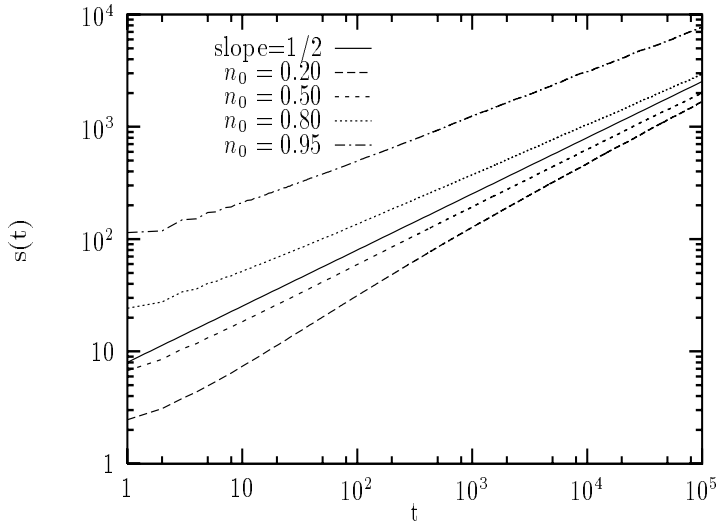

Figure 2. The length scale $s(t)$ is plotted as a function of time t. The straight line is a fit, with slope $\frac{1}{2}$.

To compute the mean interval size $s(t)$, we ran the simulation up to $t = 10^5$ time steps, and averaged the results over 100 starting distributions of particles, with the same initial density. In figure 2, we plot $s(t)$ versus t for four different values of n_0—0.2, 0.5, 0.8 and 0.95. For $n_0 = \frac{1}{2}$ we find that $s(t) \sim at^{1/z}$ with $z \simeq 1.97(1)$ and $a \simeq 5.96$, but for other values of n_0 we find that the observed value of z is different from 2. In figure 3, the running exponent $\mathrm{d}(\log s)/\mathrm{d}(\log t)$ is plotted against $1/(\log t)$ and the results show the systematic deviation away from the value $\frac{1}{2}$ expected from the scaling picture presented in the previous section. We will discuss the possible origin of this deviation later.

In figure 4, we plot the scaling function $f(x) = s(t)^2 n(k, t)$ against the scaling variable $x = k/s(t)$ for $t = 10^4$ and 3×10^4. To find the nature of the scaling function one needs to average over many configurations. This has restricted us to smaller time steps and data for three values of initial density, $n_0 = 0.2, 0.5$ and 0.8 averaged over 500, 1000 and 1500 different initial distributions of particles respectively. For all n_0, we find that the scaling function $f(x) \sim x^{-\tau}$ for $x \ll 1$ and decays exponentially for higher values of x. For $n_0 = 0.5$, we find $\tau = 1.25(1)$ in accordance with the scaling relation equation (11). For $n_0 = 0.2$ we find $\tau \simeq 1.32(2)$, while for $n_0 = 0.8$ the observed value of τ is 1.13(2). For all n_0, τ satisfies the scaling relation equation (11) if z is replaced by its effective value.

For general values of n_0, we find that the numerical values of $s(t)$ support the following form (within the time range studied):

$$s_{n_0}(t) \sim at^{1/z} + b(n_0)t^{\phi}. \tag{17}$$

The non-universal constant b is $<$, $=$ or >0 for n_0 $<$, $=$ or >0.5. To compute the prefactor b and the exponent ϕ, we plot the difference $\Delta s_{n_0}(t) = |s_{n_0}(t) - s_{1/2}(t)|$ versus t, for $n_0 = 0.2, 0.8$ and 0.95 (figure 5). The exponent ϕ is numerically found to be close to the persistence exponent $\theta = 0.375$ (table 1). In figure 6, we show the simulation data together with the fitting form equation (17) using the estimated numerical values of a and b. We find that as $n_0 \to 1$, the constant b undergoes a sharp rise so that the effective dynamical exponent of $s(t)$ is numerically close to θ for an appreciable range in time (figure 3). At the same time, we note that only the first term in equation (17) is asymptotically relevant since $\phi < \frac{1}{2}$.

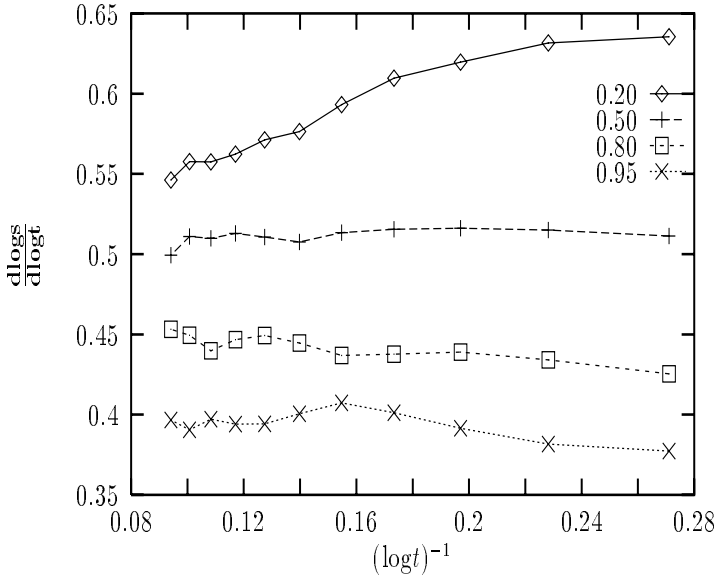

Figure 3. The effective exponent $\frac{d\log s}{d\log t}$ is plotted against $1/\log t$ for four values of starting density. For $n_0 = 0.5$, the exponent value is close to 0.5, expected from the scaling arguments. For other values of n_0, systematic deviations away from 0.5 are observed.

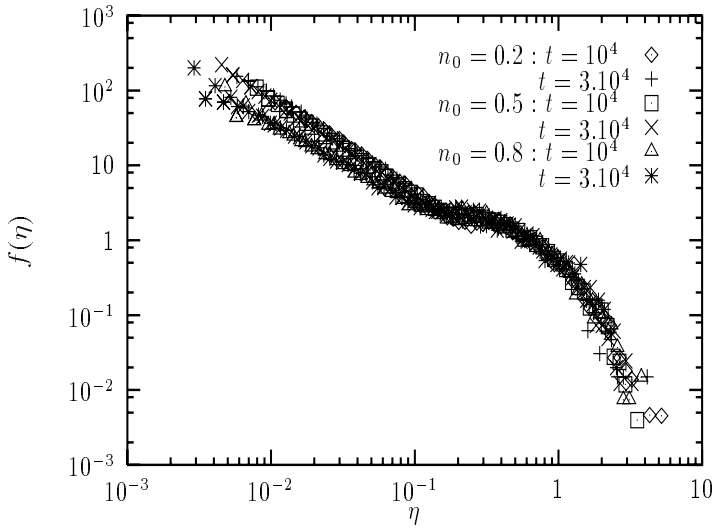

Figure 4. The scaling function $f(\eta) = s(t)^2 n(k,t)$ is plotted against the scaling variable $\eta = k/s(t)$ on a logarithmic scale. There is a power-law divergence at small η and exponential decay at large η, as predicted by the IIA calculation. The observed value of τ for $n_0 = 0.8$ is seen to be appreciably different from that for other n_0.

The two terms in equation (17) may have their origin from the two dynamical length scales in the problem, the diffusive scale $\mathcal{L}_D(t) \sim t^{1/2}$ and the persistence scale $\mathcal{L}_p(t) \sim t^{\theta}$. For large n_0, the typical interval length between two consecutive persistent sites is determined by the decay of persistence only, rather than the diffusion of the particles. So, it is understandable

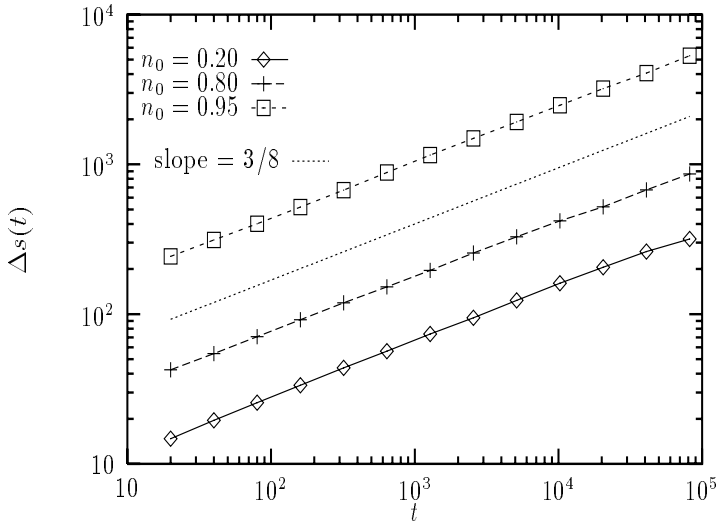

Figure 5. The difference $\Delta s(t) = |s(t) - s_{1/2}(t)|$ is plotted against t for $n_0 = 0.2$, 0.8 and 0.95. The straight line is a fit with slope $\frac{3}{8}$.

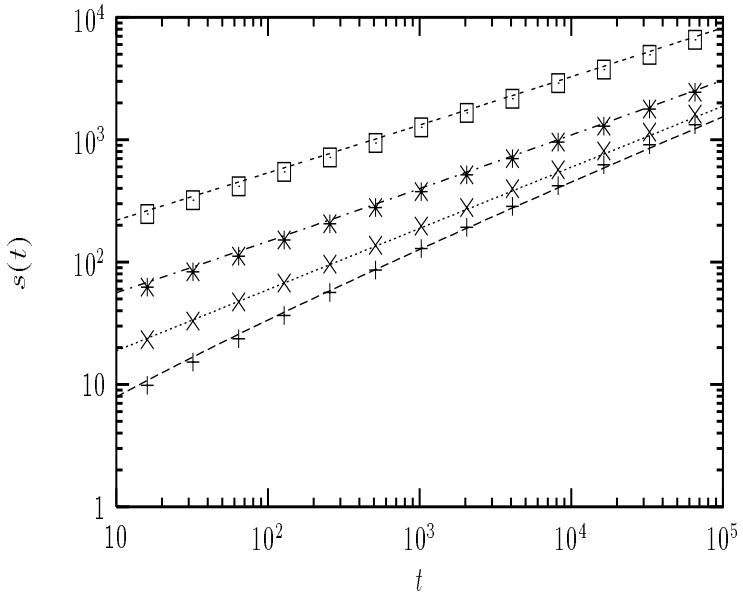

Figure 6. The observed $s(t)$ (points) with the proposed fitting form $at^{1/2} + bt^{\theta}$ (lines and curves) for $n_0 = 0.2$, 0.5, 0.8 and 0.95 (bottom to top). We have used $a = 5.96$ and the b values taken from table 1.

that the dynamical behaviour of $s(t)$ coincides with that of $\mathcal{L}_p(t)$ at least at the initial times. However at late times, when the particle density falls as a result of annihilation, the situation becomes the same as that of starting with low n_0 and the decisive scale is $\mathcal{L}_D(t)$. However, the precise form and behaviour of the prefactor $b(n_0)$ with n_0 remain to be understood.

Table 1. Results for the prefactor b and exponent ϕ as measured from simulations. The numerical value of ϕ is found to be close to the persistence exponent θ, whose exact value is 0.375. The figures in brackets represent statistical error in the last decimal place. Note the sharp rise in b as $n_0 \to 1$.

n_0	b	ϕ
0.20	−6.621	0.343 72(11)
0.50	$\simeq 0$	—
0.80	15.701	0.354 95(5)
0.95	84.672	0.365 72(4)

3. Two-point correlations

A good picture of the spatial distribution of the persistent sites and the presence of any possible correlation in their distribution is obtained from the two-point correlation $C(r, t)$, which is defined as the probability that site $\boldsymbol{x} + \boldsymbol{r}$ is persistent, given that site $\boldsymbol{x}$ is persistent (averaged over $\boldsymbol{x}$)

$$C(r, t) = \langle \rho(\boldsymbol{x}, t) \rangle^{-1} \langle \rho(\boldsymbol{x}, t) \rho(\boldsymbol{x} + \boldsymbol{r}, t) \rangle \tag{18}$$

where the brackets denote the average over the entire lattice and $\rho(\boldsymbol{x}, t)$ is the density of persistent sites: i.e. $\rho(\boldsymbol{x}, t) = 1$ if site $\boldsymbol{x}$ is persistent at time t, and 0 otherwise. Clearly, $\langle \rho(\boldsymbol{x}, t) \rangle = P(t)$ by definition.

Within the IIA, the relation between $C(r, t)$ and $n(r, t)$ (we consider $r \gg 1$, so that the discreteness of the underlying lattice can be ignored) can be written as the following infinite series:

$$C(r, t) = P(t)^{-1} n(r, t) + P(t)^{-2} \int_1^r \mathrm{d}x\, n(x, t) n(r - x, t) + P(t)^{-3} \int_1^r \mathrm{d}x\, n(x, t) \int_1^{r-x} \mathrm{d}y\, n(y, t) n(r - x - y, t) + \cdots . \tag{19}$$

The first term corresponds to the case where there is no other persistent site in the range $[0 : r]$, i.e. a single Interval of length r. The second term gives the probability that the range is split into two Intervals of length x and $r - x$ by the presence of a persistent site at x, the third term gives the probability for three Intervals and so on.

The above series can be rewritten as the following self-consistent equation for $C(r, t)$.

$$P(t) C(r, t) = n(r, t) + \int_1^r n(x, t) C(r - x, t)\, \mathrm{d}x. \tag{20}$$

In terms of the Laplace transforms $\tilde{C}(p, t) = \int_1^\infty C(r, t) \mathrm{e}^{-pr}\, \mathrm{d}r$ and $\tilde{n}(p, t) = \int_1^\infty n(s, t) \mathrm{e}^{-ps}\, \mathrm{d}s$ equation (20) becomes

$$\tilde{C}(p, t) = \frac{\tilde{n}(p, t)}{P(t) - \tilde{n}(p, t)}. \tag{21}$$

From equation (7), we find

$$\tilde{n}(p, t) = s^{-1} \tilde{f}(ps) \tag{22}$$

where $\tilde{f}(q) = \int_{s^{-1}}^\infty f(\eta) \mathrm{e}^{-q\eta}\, \mathrm{d}\eta$, which can be written in the following regularized form, using equation (9).

$$\tilde{f}(q) = s(t) P(t) - f_1(q) \tag{23}$$

where

$$f_1(q) = \int_0^\infty f(\eta)[1 - \mathrm{e}^{-q\eta}]\,\mathrm{d}\eta. \tag{24}$$

Substituting equations (22)–(24) into (21) we find that

$$\tilde{C}(p,t) = \frac{s(t)P(t)}{f_1(ps)} - 1. \tag{25}$$

The second term in the RHS can be neglected at late times, since $s(t)P(t)$ diverges as $t^{1/z-\theta}$. It follows that, in this limit, $C(r,t)$ has the dynamic scaling form

$$C(r,t) = P(t)g\left(\frac{r}{s(t)}\right) \tag{26}$$

where

$$\tilde{g}(q) = \frac{1}{f_1(q)} \tag{27}$$

is the Laplace transform of $g(x)$: $\tilde{g}(q) = \int_0^\infty g(x)\mathrm{e}^{-qx}\,\mathrm{d}x$.

The preceding expressions can be used to deduce the limiting behaviour of the scaling function $g(\eta)$ for the cases $\eta \ll 1$ and $\eta \gg 1$, without needing to solve equation (19) or (20) explicitly.

Case 1. $\eta \gg 1$.

To find the asymptotic behaviour of $g(\eta)$, we note that $f_1(q)$ vanishes near $q = 0$ as $f_1(q) \sim q$. Thus $\tilde{g}(q) \sim \frac{1}{q}$ as $q \to 0$ from equation (27). By standard results in the theory of Laplace transforms [14], this implies that $g(\eta) \sim 1$ as $\eta \to \infty$.

Case 2. $\eta \ll 1$.

To analyse this case, consider the real-space relation equation (19). For $\eta \ll 1$, or, equivalently, $r \ll s$, we have shown that $n(r,t) \sim P(t)r^{-\tau}$. It is clear that, in this range, the RHS of equation (19) is time independent, so $C(r,t)$ in the LHS should also be time independent. From the dynamic scaling form equation (26), we find that this is possible only if the scaling function is a power law near the origin: $g(\eta) \sim \eta^{-\alpha}$ as $\eta \to 0$. After substituting in equation (26) and requiring the resulting expression to be time independent, we find

$$\alpha = z\theta. \tag{28}$$

We find $C(r,t) \sim r^{-\alpha}$ for $r \ll s$ and $C(r,t) \simeq P(t)$ for $r \gg s$. The power law decay at small distances is expected, because the RHS of equation (19) contains only scale invariant terms in this limit, hence the LHS also should be likewise. In appendix B, we show that this is also consistent with equation (20).

We see that, in the IIA calculation, the length scale $s(t)$ demarcates the correlated and uncorrelated regions for $C(r,t)$. In the correlated region ($r \ll s(t)$), the persistent sites form a fractal with fractal dimension $d_f = d - \alpha = \frac{1}{4}$, with the correlation length $s(t)$ increasing with time as $s \sim t^{1/2}$. The IIA results agree very well with those of numerical simulations [10], showing the validity of the approximation.

4. Conclusion

Persistent sites are shown to have strong correlations in their spatial distribution. In the one-dimensional $A + A \to \emptyset$ reaction–diffusion system, we show that there is a length scale

$s(t)$, diverging with time as $s(t) \sim t^{1/z}$, which demarcates the correlated region from the uncorrelated one. We argue that $z = 2$ in the large-t limit. Persistent sites separated by distance $k \ll s(t)$ are highly unlikely to have a particle A between them and so retain their persistent character. Only persistent sites separated by distance $\gg s(t)$ take part in the decay of persistence at subsequent times.

We find that if k is the distance of separation between any two consecutive persistent sites, then for $k \ll s(t)$ the distribution of k is scale free and decays algebraically as $k^{-\tau}$ with $\tau = 2 - z\theta$. We show this using the IIA, which assumes no correlation in the lengths of any two adjacent intervals. We have verified our results by numerical simulations, which suggests the validity of the IIA. Under the IIA, our calculation for the two-point correlation shows that over length scales $r \ll s(t)$, the persistent site distribution over the lattice is a fractal with dimension $d_f = \tau - 1$, in accordance with our earlier observations [10].

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Appendix A

The divergence in the first integral in equation (10) can be separated out as follows. We write $f(\eta - x) = f(\eta) + \Delta_x f(\eta)$ and $\beta(\eta - x) = \beta(\eta) + \Delta_x \beta(\eta)$, so that $\lim_{x\to 0} \Delta_x f(\eta) = \lim_{x\to 0} \Delta_x \beta(\eta) = 0$.

After substituting for $f(\eta - x)$ and $\beta(\eta - x)$, the divergent part of the integral separates into the following terms:

$$\int_{s(t)^{-1}}^{\frac{\eta}{2}} f(x) f(\eta - x)\beta(\eta - x)\,\mathrm{d}x = f(\eta)\beta(\eta)\int_{s(t)^{-1}}^{\frac{\eta}{2}} f(x)\,\mathrm{d}x + f(\eta)\int_0^{\frac{\eta}{2}} f(x)\Delta_x\beta(\eta)\,\mathrm{d}x$$
$$+\beta(\eta)\int_0^{\frac{\eta}{2}} f(x)\Delta_x f(\eta)\,\mathrm{d}x + \int_0^{\frac{\eta}{2}} f(x)\Delta_x f(\eta)\Delta_x\beta(\eta)\,\mathrm{d}x.$$

The first term is divergent near the origin, while all other terms are finite by construction. Now we rewrite the first term using the equality $\int_{s(t)^{-1}}^{\infty} f(x)\,\mathrm{d}x = s(t)P(t)$. After some simplifications, the integral becomes

$$\int_{s(t)^{-1}}^{\frac{\eta}{2}} f(x) f(\eta - x)\beta(\eta - x)\,\mathrm{d}x = f(\eta)\beta(\eta)s(t)P(t)$$
$$+\int_0^{\frac{\eta}{2}} f(x) f(\eta - x)[\beta(\eta - x) - \beta(\eta)]\,\mathrm{d}x$$
$$+\int_0^{\frac{\eta}{2}} f(x)[f(\eta - x) - f(\eta)] - \beta(\eta) f(\eta)\int_{\frac{\eta}{2}}^{\infty} f(x)\,\mathrm{d}x.$$

The first term is the divergent part of the integral, which exactly cancels the last term in equation (10), to give the regularized equation (12).

Appendix B

For $r \gg 1$, it is reasonable to assume that the higher-order terms in the RHS of equation (19) will contribute more than the first term, i.e. the range $[0 : r]$ is more likely to be covered

with more than one Interval than a single one of length r. After using this approximation, and substituting $n(r,t) \simeq (\tau-1)P(t)r^{-\tau}$ in the continuum limit, equation (20) is simplified to

$$C(r,t) \simeq (\tau-1)\int_1^{r-1}(r-x)^{-\tau}C(x,t)\,\mathrm{d}x.$$

Our purpose is to see whether the equation

$$r^{-\alpha} \simeq (\tau-1)\int_1^{r-1}x^{-\alpha}(r-x)^{-\tau}\,\mathrm{d}x \tag{B.1}$$

is consistent for $\alpha = z\theta = 2-\tau$ (equation (11)) at $r \gg 1$.

The integral $I = \int_1^{r-1}x^{-\tau}(r-x)^{-\alpha}\,\mathrm{d}x$ can be transformed by change of variables into the more standard form [15]

$$\int_0^{r-2}(1+y)^{-\tau}[r-1-y]^{-\alpha}\,\mathrm{d}y \simeq \frac{r^{1-\alpha}}{1-\alpha} \times F(1,\tau;2-\alpha;-r) \qquad \text{for} \quad \alpha<1 \qquad \text{and} \qquad r \gg 1$$

where $F(a,b;c;z)$ is the Gauss hypergeometric function. For $b=c$, $F(a,b;b;z) = (1-z)^{-a}$ exactly, independent of b [16]. Thus, for $\alpha = 2-\tau$ we find

$$(\tau-1)I = r^{-\alpha}\left[1+\mathrm{o}\left(\frac{1}{r}\right)\right]$$

which is consistent with equation (B.1), at $r \gg 1$.

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